Field theory amplitudes in a space with SU(2) fuzziness

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Abstract

The structure of transition amplitudes in field theory in a three-dimensional space whose spatial coordinates are noncommutative and satisfy the SU(2) Lie algebra commutation relations is examined. In particular, the basic notions for constructing the observables of the theory as well as subtleties related to the proper treatment of δ distributions (corresponding to conservation laws) are introduced. Explicit examples are given for scalar field theory amplitudes in the lowest order of perturbation.

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1 Introduction

Recently much attention has been paid to the formulation and study of field theories on noncommutative spaces. The motivation is partly the natural appearance of noncommutative spaces in some areas of physics, a recent one occurring in string theory. In particular, it has become clear that the longitudinal directions of D-branes in the presence of a constant B-field background appear to be noncommutative, as seen by the ends of open strings [1–4]. In this case the coordinates satisfy the canonical relation

$$[\widehat{x}_a, \widehat{x}_b] = \mathrm{i}\,\theta_{a\,b}\,\mathbf{1}, \tag{1}$$

in which θ is an antisymmetric constant tensor and $\mathbf{1}$ represents the unit operator. Although due to the presence of the background field, it might seem as if a Poincaré invariant interpretation of field theories on canonical noncommutative spaces is not possible, it has been shown that a twisted version of Poincaré symmetry can be introduced as the alternative symmetry of field theories on canonical spaces [5]. The theoretical and phenomenological implications of possible noncommutative coordinates have extensively been studied [6].

One direction to extend studies on noncommutative spaces is to consider spaces for which the commutators of the coordinates are not constants. Examples of this kind are the cases with a q-deformed plane and noncommutative cylinder ($S^1 \times \mathbb{R}$) [7]. It is shown that, while the ultraviolet (UV) behavior of the theory in a q-deformed case is worse than an ordinary plane, the theory on a noncommutative cylinder, contrary to its commutative version, appears to be UV-finite [7]. Another example of this kind is the so called κ-Poincaré algebra, in which the noncommutativity is introduced between spatial directions and time, that is [8, 9]

$$\begin{aligned}[\widehat{x}_a, \widehat{t}\,] &= \frac{\mathrm{i}}{\kappa}\,\widehat{x}_a,\\ [\widehat{x}_a, \widehat{x}_b] &= 0,\end{aligned} \tag{2}$$

where κ is a constant. The formulation of quantum field theories on this kind of spaces has been studied in [10, 11].

In the noncommutative cylinder and κ-Poincaré cases mentioned above the noncommutativity is involved by the time direction. Other interesting examples are the models in which the (dimensionless) *spatial* positions operators satisfy the commutation relations of a Lie algebra [12, 13]:

$$[\widehat{x}_a, \widehat{x}_b] = f^c{}_{a\,b}\,\widehat{x}_c, \tag{3}$$

where the $f^c{}_{a\,b}$'s are structure constants of a Lie algebra. One example of this kind is the algebra SO(3), or SU(2). A special case of this is the so called fuzzy sphere [14, 15], where an irreducible representation of the position operators is used that makes the Casimir operator of the algebra, $(\widehat{x}_1)^2 + (\widehat{x}_2)^2 + (\widehat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One

can consider the square root of this Casimir operator as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere). Different aspects of field theories on the fuzzy sphere, including the fate of the UV-divergences of the Euclidean theory, the structure of UV/IR mixing, as well as topologically nontrivial field configurations have already been examined [16–18].

In a previous work [19] a model was introduced in which the representation was not restricted to an irreducible one; instead the whole group was employed. In particular, the regular representation of the group, which contains all representations, was considered. As a consequence in such models one is dealing with the whole space, rather than a 2-dimensional sub-space as in the case of fuzzy sphere. The space of the corresponding momenta is an ordinary (commutative) space and is compact if and only if the group is compact. In fact, one can consider the momenta as the coordinates of the group. So a by-product of such a model would be the elimination of any UV-divergence in any field theory constructed on such a space. One important implication of the elimination of the ultraviolet divergences, as we shall see in more detail later, would be that there will not remain place for the so called UV/IR mixing effect [20], which is known as a common phenomenon one expects to be going to face in a models with canonical noncommutativity, the algebra (1). In [19] the basic ingredients for calculus on a linear fuzzy space, together with basic notions for a field theory on such a space, including Lagrangian and elements for a perturbation theory, were introduced. The models based on the regular representations of SU(2) and SO(3) were treated in more detail, giving the explicit form of the tools and notions introduced in their general form.

In the present work the aim is to examine the structure of amplitudes coming from a field theory based on a space with SU(2) fuzziness. In particular, we introduce the basic elements by which one can compute the matrix elements corresponding to the transition between initial and final states. The contribution entailed in a perturbative expansion of the amplitudes are presented in the lowest order (tree level) for a self-interacting scalar field theory.

The scheme of this paper is the following. In section 2, a brief review is given of the calculus and field theory on noncommutative spaces of the Lie algebra type, so that the present paper is self-contained. In section 3 the basic elements of transition matrix elements are introduced and discussed. Explicit examples are presented to show how thiese things work. Section 4 is devoted to our conclusion.

2 Basic notions

2.1 Calculational tools

For a compact group G, there is a unique measure $\mathrm{d}U$ (up to a multiplicative constant) with the invariance properties

$$\begin{aligned} \mathrm{d}(V\,U) &= \mathrm{d}U, \\ \mathrm{d}(U\,V) &= \mathrm{d}U, \\ \mathrm{d}(U^{-1}) &= \mathrm{d}U, \end{aligned} \tag{4}$$

for an arbitrary element V of the group. These mean that this measure is invariant under the left-translation, right-translation, and inversion. This measure, the (left-right-invariant) Haar measure, is unique up to a normalization constant, which defines the volume of the group:

$$\int_G \mathrm{d}U = \mathrm{vol}(G). \tag{5}$$

Using this measure, one constructs a vector space as follows. Corresponding to each group element U an element $\mathfrak{e}(U)$ is introduced, and the elements of the vector space are linear combinations of these elements:

$$f := \int \mathrm{d}U\; f(U)\, \mathfrak{e}(U), \tag{6}$$

The group algebra is this vector space, equipped with the multiplication

$$f\,g := \int \mathrm{d}U\, \mathrm{d}V\; f(U)\, g(V)\, \mathfrak{e}(U\,V), \tag{7}$$

where $(U\,V)$ is the usual product of the group elements. $f(U)$ and $g(U)$ belong to a field (here the field of complex numbers). It can be seen that if one takes the central extension of the group $\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$, the so-called Heisenberg group, with the algebra (1), the above definition results in the well-known star product of two functions, provided f and g are interpreted as the Fourier transforms of the functions.

So there is a correspondence between functionals defined on the group, and the group algebra. The definition (7) can be rewritten as

$$\begin{aligned} (f\,g)(W) &= \int \mathrm{d}V\; f(W\,V^{-1})\, g(V), \\ &= \int \mathrm{d}U\; f(U)\, g(U^{-1}\,W). \end{aligned} \tag{8}$$

Using Schur's lemmas, one proves the so called grand orthogonality theorem, which states that there is an orthogonality relation between the matrix functions of the group:

$$\int \mathrm{d}U\; {U_\lambda}^a{}_b\, {U_\mu^{-1}}{}^c{}_d = \frac{\mathrm{vol}(G)}{\dim_\lambda}\, \delta_{\lambda\,\mu}\, \delta^a_d\, \delta^c_b, \tag{9}$$

where U_λ is the matrix of the element U of the group in the irreducible representation λ, and $\dim_\lambda$ is the dimension of the representation λ. Exploiting the unitarity of these representations, one can write (9) in the more familiar form

$$\int \mathrm{d}U \; {U_\lambda}^a{}_b \, {U^*_\mu}_d{}^c = \frac{\mathrm{vol}(G)}{\dim_\lambda} \, \delta_{\lambda\,\mu} \, \delta^a_d \, \delta^c_b. \tag{10}$$

Using this orthogonality relation, one can obtain an orthogonality relation between the characters of the group:

$$\int \mathrm{d}U \; \chi_\lambda(U) \, \chi_\mu(U^{-1}) = \mathrm{vol}(G) \, \delta_{\lambda\,\mu}, \tag{11}$$

or

$$\int \mathrm{d}U \; \chi_\lambda(U) \, \chi^*_\mu(U) = \mathrm{vol}(G) \, \delta_{\lambda\,\mu}, \tag{12}$$

where

$$\chi_\lambda(U) := {U_\lambda}^a{}_a. \tag{13}$$

The delta distribution is defined through

$$\int \mathrm{d}U \; \delta(U) \, f(U) := f(\mathbf{1}), \tag{14}$$

where $\mathbf{1}$ is the identity element of the group; we notice that as usual the delta picks up the value of the function at the origin, $U = \mathbf{1}$. It is easy to see that this delta distribution is invariant under similarity transformations, as well as inversion of the argument:

$$\begin{aligned} \delta(V\,U\,V^{-1}) &= \delta(U), \\ \delta(U^{-1}) &= \delta(U). \end{aligned} \tag{15}$$

The first relation shows that if the argument of the delta is a product of group elements, then any cyclic permutation of these elements leaves the delta unchanged.

The regular representation of the group is defined through

$$U_{\mathrm{reg}} \, \mathfrak{e}(V) := \mathfrak{e}(U\,V), \tag{16}$$

from which it is seen that the matrix element of this linear operator is

$$U_{\mathrm{reg}}(W, V) = \delta(W^{-1}\,U\,V). \tag{17}$$

This shows that the trace of the regular representation is proportional to the delta distribution:

$$\begin{aligned} \chi_{\mathrm{reg}}(U) = &\int \mathrm{d}V \; U_{\mathrm{reg}}(V, V), \\ =&\mathrm{vol}(G) \, \delta(U). \end{aligned} \tag{18}$$

So the delta distribution can be expanded in terms of the matrix functions (in fact in terms of the characters of irreducible representations). The result is

$$\delta(U) = \sum_\lambda \frac{\dim_\lambda}{\mathrm{vol}(G)}\,\chi_\lambda(U), \tag{19}$$

or

$$\begin{aligned}\delta(U\,V^{-1}) &= \sum_\lambda \frac{\dim_\lambda}{\mathrm{vol}(G)}\,{U_\lambda}^a{}_b\,{V_\lambda^{-1}}^b{}_a,\\ &= \sum_\lambda \frac{\dim_\lambda}{\mathrm{vol}(G)}\,{U_\lambda}^a{}_b\,{V^*_\lambda}_a{}^b.\end{aligned} \tag{20}$$

This shows that the other functions are also expandable in terms of the matrix functions:

$$f(U) = \sum_\lambda \frac{\dim_\lambda}{\mathrm{vol}(G)}\,{U_\lambda}^a{}_b\,{f_\lambda}_a{}^b, \tag{21}$$

where

$$\begin{aligned}{f_\lambda}_a{}^b &:= \int \mathrm{d}V\;{V_\lambda^{-1}}^b{}_a\,f(V),\\ &= \int \mathrm{d}V\;{V^*_\lambda}_a{}^b\,f(V).\end{aligned} \tag{22}$$

Using this and (8), one arrives at

$$(f\,g)_{\lambda a}{}^b = {f_\lambda}_a{}^c\,{g_\lambda}_c{}^b. \tag{23}$$

Next, one can define an inner product on the group algebra. Defining

$$\langle \mathfrak{e}(U), \mathfrak{e}(V)\rangle := \delta(U^{-1}\,V), \tag{24}$$

and demanding that the inner product be linear with respect to its second argument and antilinear with respect to its first argument, one arrives at

$$\begin{aligned}\langle f, g\rangle &= \int \mathrm{d}U\;f^*(U)\,g(U),\\ &= \sum_\lambda \frac{\dim_\lambda}{\mathrm{vol}(G)}\,{f^*_\lambda}^a{}_b\,{g_\lambda}_a{}^b.\end{aligned} \tag{25}$$

Finally, one defines a star operation through

$$f^\star(U) := f^*(U^{-1}). \tag{26}$$

This is in fact equivalent to definition of the star operation in the group algebra by

$$[\mathfrak{e}(U)]^\star := \mathfrak{e}(U^{-1}). \tag{27}$$

It is then easy to see that

$$(f\,g)^\star = g^\star\,f^\star, \tag{28}$$

$$\langle f, g\rangle = (f^\star\,g)(\mathbf{1}). \tag{29}$$

Here a note is in order. While the results of this section were obtained for compact groups, in some cases compactness is not necessary. It is easy to see that, provided (4) holds, (6)-(8), (14)-(17), (24), the first equality in (25), and (26)-(29) are still true, even if the group is noncompact.

2.2 Field theory

Based on the calculational tools presented in the previous subsection, here we can present the construction of a field theory on a noncommutative space, the commutation relations of which are those of a compact Lie group. In this work we consider the simplest case: the scalar theory. To avoid explicit calculus on such a noncommutative space, everything is defined on the momentum space. This space is commutative and one can attribute well-defined (local) coordinates to it, so that ordinary differential and integral calculus (on manifolds) can be performed on it. As far as observables of field theories are concerned, this momentum representation is sufficient.

To give motivation for the particular form of the action that is going to be written for a real scalar field, we first consider the real scalar field on an ordinary $\mathbb{R}^D$ space. To be consistent with the notation used throughout this paper, the Fourier transform (only on space) of the field is denoted by ϕ, while the field itself is denoted by $\tilde{\phi}$. So we have

$$\tilde{\phi}(\mathbf{r}) = \int \frac{\mathrm{d}^D k}{(2\,\pi)^D}\;\phi(\mathbf{k})\;\exp(\mathrm{i}\,\mathbf{r}\cdot\mathbf{k}). \tag{30}$$

An action for a scalar field is

$$S = \int \mathrm{d}t\,\mathrm{d}^D r\;\left\{\frac{1}{2}\,\left[\dot{\tilde{\phi}}(\mathbf{r})\,\dot{\tilde{\phi}}(\mathbf{r}) + \tilde{\phi}(\mathbf{r})\,\tilde{O}(\nabla)\,\tilde{\phi}(\mathbf{r})\right] - \sum_{j=3}^{n}\frac{g_j}{j!}\,[\tilde{\phi}(\mathbf{r})]^j\right\}, \tag{31}$$

where the g_j are constants and $\tilde{O}(\nabla)$ is a differential operator. This action is translation invariant, that is, invariant under the transformations

$$\tilde{\phi}(\mathbf{r}) \to \tilde{\phi}'(\mathbf{r}) := \tilde{\phi}(\mathbf{r}-\mathbf{a}), \tag{32}$$

where $\mathbf{a}$ is constant.

One can write the action (31) and the transformation (32) in terms of the

Fourier transforms:

$$S=\int \mathrm{d}t\,\left\{\frac{1}{2}\int\frac{\mathrm{d}^D k_1\,\mathrm{d}^D k_2}{(2\,\pi)^{2\,D}}\,[\dot\phi(\mathbf{k}_1)\,\dot\phi(\mathbf{k}_2)\right.$$
$$+\phi(\mathbf{k}_1)\,O(\mathbf{k}_2)\,\phi(\mathbf{k}_2)]\,[(2\,\pi)^D\,\delta(\mathbf{k}_1+\mathbf{k}_2)]$$
$$\left.-\sum_{j=3}^{n}\frac{g_j}{j!}\int\left[\prod_{l=1}^{j}\frac{\mathrm{d}^D k_l\;\phi(\mathbf{k}_l)}{(2\,\pi)^D}\right]\,[(2\,\pi)^D\,\delta(\mathbf{k}_1+\cdots+\mathbf{k}_j)]\right\},\quad (33)$$

and

$$\phi(\mathbf{k})\to\phi'(\mathbf{k}):=\exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{a})\,\phi(\mathbf{k}).\quad (34)$$

Considering the space of the $\mathbf{k}$ as a group ($\mathbb{R}^D$), one notices that $(\mathrm{d}^D k)/(2\,\pi)^D$ is the measure of this group that is invariant under right translation, left translation, and inversion. It is not normalizable in the sense of (5), as this group is not compact. One also notices that $\exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{a})$ is nothing but the representation $\mathbf{a}$ of the group element corresponding to the coordinates $\mathbf{k}$. As this representation is one dimensional, $\exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{a})$ is also the determinant of this representation.

Now we come to the case on fuzzy space. A real scalar field ϕ is defined as a real member of the group algebra:

$$\phi^\star=\phi.\quad (35)$$

In analogy with the action on ordinary space, one may suggest the action

$$S=\int \mathrm{d}t\left\{\frac{1}{2}\int \mathrm{d}U_1\,\mathrm{d}U_2\,\left[\dot\phi(U_1)\,\dot\phi(U_2)+\int \mathrm{d}U\;\phi(U_1)\,O(U_2,U)\,\phi(U)\right]\,\delta(U_1\,U_2)\right.$$
$$\left.-\sum_{j=3}^{n}\frac{g_j}{j!}\int\left[\prod_{l=1}^{j}\mathrm{d}U_l\;\phi(U_l)\right]\,\delta(U_1\cdots U_j)\right\}.\quad (36)$$

where g_j are constants and O is a linear operator from the group algebra to the group algebra. For the action on the ordinary space, one has

$$O(\mathbf{k}_2,\mathbf{k})\propto\delta(\mathbf{k}_2-\mathbf{k}).\quad (37)$$

In analogy with that, we take

$$O(U_2,U)=O(U)\,\delta(U_2\,U^{-1}).\quad (38)$$

From now on, it is assumed that this is the case. So

$$S=\int \mathrm{d}t\left\{\frac{1}{2}\int \mathrm{d}U_1\,\mathrm{d}U_2\,\left[\dot\phi(U_1)\,\dot\phi(U_2)+\phi(U_1)\,O(U_2)\,\phi(U_2)\right]\,\delta(U_1\,U_2)\right.$$
$$\left.-\sum_{j=3}^{n}\frac{g_j}{j!}\int\left[\prod_{l=1}^{j}\mathrm{d}U_l\;\phi(U_l)\right]\,\delta(U_1\cdots U_j)\right\},\quad (39)$$

A simple choice for O is

$$O(U) = c\,\chi_\lambda(U + U^{-1} - 2\,\mathbf{1}) - m^2, \tag{40}$$

where λ is a representation of the group, and c and m are constants. An argument for the plausibility of this choice is the following. Consider a Lie group and a group element near its identity, so that

$$\begin{aligned} U_\lambda = {}& \exp(\tilde{k}^a\, T_{a\,\lambda}), \\ \approx {}& \mathbf{1}_\lambda + \tilde{k}^a\, T_{a\,\lambda} + \frac{1}{2}\,(\tilde{k}^a\, T_{a\,\lambda})^2, \end{aligned} \tag{41}$$

where T_a are the generators of the group. One has

$$O(U) \approx c\,\chi_\lambda(T_a\, T_b)\, \tilde{k}^a\, \tilde{k}^b - m^2, \tag{42}$$

which is a constant plus a bilinear form in $\tilde{\mathbf{k}}$, just as was expected for an ordinary scalar field. In fact, if one introduces a small constant ℓ so that $\tilde{k}$ is proportional to ℓ, and c is proportional to ℓ^{-2}, then in the limit $\ell \to 0$ the expression (42) is exactly equal to a constant plus a bilinear form.

An action of the form (39) with the choice (40) also has a symmetry under

$$\phi(U) \to \phi(V\, U\, V^{-1}), \tag{43}$$

where V is an arbitrary member of the group.

One can write the action (39) in terms of the Fourier transform of the field in time:

$$\phi(t, U) =: \int \frac{\mathrm{d}\omega}{2\,\pi}\ \exp(-\mathrm{i}\,\omega\, t)\, \check{\phi}(\omega, U), \tag{44}$$

to arrive at

$$\begin{aligned} S = {}& \frac{1}{2} \int \frac{\mathrm{d}\omega_1\, \mathrm{d}U_1}{2\,\pi}\, \frac{\mathrm{d}\omega_2\, \mathrm{d}U_2}{2\,\pi}\ \left[-\omega_1\,\omega_2\, \check{\phi}(U_1)\, \check{\phi}(U_2) + \check{\phi}(U_1)\, O(U_2)\, \check{\phi}(U_2)\right] \\ & \times [2\,\pi\, \delta(\omega_1 + \omega_2)\, \delta(U_1\, U_2)] \\ & - \sum_{j=3}^{n} \frac{g_j}{j!} \int \left[\prod_{l=1}^{j} \frac{\mathrm{d}\omega_l\, \mathrm{d}U_l}{2\,\pi}\ \check{\phi}(U_l)\right] [2\,\pi\, \delta(\omega_1 + \cdots + \omega_j)\, \delta(U_1 \cdots U_j)]. \end{aligned} \tag{45}$$

The first two terms represent a free action, with the propagator

$$\check{\Delta}(\omega, U) := \frac{\mathrm{i}\,\hbar}{\omega^2 + O(U)}. \tag{46}$$

Putting the denominator of this propagator equal to zero gives the relation between ω and U for free particles (the mass-shell condition). The third term contains interactions. Any Feynman graph would consist of propagators and j-line vertices to which one assigns

$$V_j := \frac{g_j}{\mathrm{i}\,\hbar\, j!}\, 2\,\pi\, \delta(\omega_1 + \cdots + \omega_j) \sum_{\Pi} \delta(U_{\Pi(1)} \cdots U_{\Pi(j)}), \tag{47}$$

where the summation runs over all j-permutations. In practice, as we will see later, due to cyclic symmetry of arguments of the δ functions mentioned earlier, permutations that are different up to a cyclic change just come in the sum with a proper weight. Also, for any internal line there is an integration over U and ω, with the measure $\mathrm{d}\omega\,\mathrm{d}U/(2\,\pi)$. As the group is assumed to be compact, the integration over the group is integration over a compact volume. Hence there would be no UV-divergences.

It is worth to mention a crucial difference between the way that δ functions appear in our model and in models defined on ordinary spaces. Here, as mentioned above, each possible ordering of legs of a vertex comes with a different δ, except the cases that two orderings are different up to a cyclic permutation. This is in contrast to models on ordinary space, in which all possible orderings have the common factor of one single $\delta\big(\sum \mathbf{k}_i\big)$, representing the momentum conservation in that vertex.

Similar to the above observation, δ functions appear in theories defined on κ-deformed spaces, as pointed out in the Introduction . In these theories, the ordinary summation of momenta in each vertex is replaced with a new summation rule, occasionally called a dotted sum ($\dot{+}$) [10]. This new sum, contrary to an ordinary sum, is non-Abelian, and as a consequence, the δ function coming with each possible ordering of the legs are different [10, 11].

One can compare this model to a field theory on a group manifold. In the latter model, the integration in (36) or (39) would be over the position, not over the momenta, and the operator O would be differentiation with respect to the coordinates. In a model on a group manifold, the position coordinates are still commuting but the momenta are not. Here the situation is reversed, and this is not only a matter of convenience. The operator O determines which model is being investigated: it is algebraic in terms of the momenta and differentiation in terms of the position. For models on group manifolds with compact groups, there would be no infrared (IR) divergences while here there is no UV-divergence. The fact that for a noncommutative geometry based on the Lie groups the momenta are still commuting is the reason that here the momentum picture has been preferred to the position picture.

2.3 An example: the group SU(2)

For the group SU(2), one has

$$f^a{}_{b\,c} = \epsilon^a{}_{b\,c}. \tag{48}$$

A group element U can be characterized by the coordinates (k^1, k^2, k^3) such that

$$U = \exp(\ell\, k^a\, T_a), \tag{49}$$

where ℓ is a constant. The invariant measure is

$$\mathrm{d}U = \frac{\sin^2(\ell\, k/2)}{(\ell\, k/2)^2}\,\frac{\mathrm{d}^3 k}{(2\,\pi)^3}, \tag{50}$$

where

$$k := \left(\delta_{a\,b}\,k^a\,k^b\right)^{1/2}. \tag{51}$$

The reason for this particular choice of normalization is that for small values of k, (50) reduces to the integration measure corresponding to the ordinary space. The integration region for the coordinates is

$$k \le \frac{2\,\pi}{\ell}. \tag{52}$$

In the small-k limit, one also has

$$\delta(U_1\ \cdots\ U_l) \approx (2\,\pi)^3\,\delta^3(\mathbf{k}_1 + \cdots + \mathbf{k}_l), \tag{53}$$

which ensures an approximate momentum conservation. The exact conservation law, however, is that at each vertex the product of incoming group elements should be unity. For the case of a three-leg vertex, one can write this condition as

$$\exp(\ell\,k_1^a\,T_a)\ \exp(\ell\,k_2^a\,T_a)\ \exp(\ell\,k_3^a\,T_a) = 1, \tag{54}$$

or a similar condition in which $\mathbf{k}_1$ is replaced by $\mathbf{k}_2$ and vice versa. One has

$$\exp(\ell\,k_1^a\,T_a)\ \exp(\ell\,k_2^a\,T_a) =: \exp[\ell\,\gamma^a(\mathbf{k}_1,\mathbf{k}_2)\,T_a], \tag{55}$$

where the function $\boldsymbol{\gamma}$ enjoys the properties

$$\begin{aligned}
\boldsymbol{\gamma}[\mathbf{k}_1,\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_3)] =&\boldsymbol{\gamma}[\boldsymbol{\gamma}(\mathbf{k}_1,\mathbf{k}_2),\mathbf{k}_3], &(56)\\
\boldsymbol{\gamma}(-\mathbf{k}_1,-\mathbf{k}_2) =&-\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_1), &(57)\\
\boldsymbol{\gamma}(\mathbf{k},-\mathbf{k}) =&0. &(58)
\end{aligned}$$

Therefore, (54) becomes one of the three equivalent forms

$$\begin{aligned}
\mathbf{k}_3 &= -\,\boldsymbol{\gamma}(\mathbf{k}_1,\mathbf{k}_2),\\
\mathbf{k}_2 &= -\,\boldsymbol{\gamma}(\mathbf{k}_3,\mathbf{k}_1),\\
\mathbf{k}_1 &= -\,\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_3).
\end{aligned} \tag{59}$$

The explicit form of $\boldsymbol{\gamma}$ is obtained from

$$\begin{aligned}
\cos\frac{\ell\,\gamma}{2} =&\ \cos\frac{\ell\,k_1}{2}\ \cos\frac{\ell\,k_2}{2} - \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1\,k_2}\ \sin\frac{\ell\,k_1}{2}\ \sin\frac{\ell\,k_2}{2},\\
\frac{\gamma^a}{\gamma}\ \sin\frac{\ell\,\gamma}{2} =&\ \epsilon^a{}_{b\,c}\,\frac{k_1^b\,k_2^c}{k_1\,k_2}\ \sin\frac{\ell\,k_1}{2}\ \sin\frac{\ell\,k_2}{2}\\
&+\frac{k_1^a}{k_1}\ \sin\frac{\ell\,k_1}{2}\ \cos\frac{\ell\,k_2}{2} + \frac{k_2^a}{k_2}\ \sin\frac{\ell\,k_2}{2}\ \cos\frac{\ell\,k_1}{2}.
\end{aligned} \tag{60}$$

It is easy to see that in the limit $\ell \to 0$, $\boldsymbol{\gamma}$ tends to $\mathbf{k}_1 + \mathbf{k}_2$, as expected.

The choice (40) for O turns to be

$$O = 2\,c\left\{\frac{\sin\left[\left(s+\frac{1}{2}\right)\ \ell\,k\right]}{\sin\dfrac{\ell\,k}{2}} - (2\,s+1)\right\} - m^2, \tag{61}$$

where s is the spin of the representation. For small values of k, this is turned to

$$O \approx -c\,\frac{s\,(s+1)\,(2\,s+1)}{3}\,(\ell k)^2 - m^2, \qquad (\ell\,k) \ll 1. \tag{62}$$

One chooses c so that in the small-k limit O takes the ordinary form of the propagator inverse:

$$O \approx -k^2 - m^2, \qquad (\ell\,k) \ll 1. \tag{63}$$

Choosing

$$c = \frac{3}{s\,(s+1)\,(2\,s+1)\,\ell^2}, \tag{64}$$

the propagator becomes

$$\check{\Delta}(\omega, \mathbf{k}) = \frac{\mathrm{i}\,\hbar}{\omega^2 + \dfrac{6}{s\,(s+1)\,(2\,s+1)\,\ell^2}\left\{\dfrac{\sin\left[\left(s+\dfrac{1}{2}\right)\,\ell\,k\right]}{\sin\dfrac{\ell\,k}{2}} - (2\,s+1)\right\} - m^2}. \tag{65}$$

It is easy to see that in the limit $\ell \to 0$, the usual commutative propagator is recovered.

Similar things hold for the group SO(3). One only has to replace the integration region by

$$k \le \frac{\pi}{\ell}. \tag{52$'$}$$

A consequence of the compactness of the momentum space is that field theories based of spaces with Lie group fuzziness corresponding to compact groups are free from UV-divergences. The above restriction on the integration region in momentum space, as well as the UV-finiteness of theory, are very similar to those one has in theories defined on lattices. This would be no surprise for this behavior once one mentions that the eigenvalues of the space coordinates are discrete as a consequence of the coordinates satisfying the SU(2) or SO(3) algebras, and in general that of a compact Lie group. There are, however, differences between such theories and theories based on space lattices: In the latter theories there are no continuous space symmetries, while in the former one there are (rotation in the case of SO(3) or SU(2)); in the former case it is not possible to determine all position operators simultaneously, while in the latter case it is; and in the latter case the positions are discrete, while in the former case the position eigenvalues are discrete.

The UV-finiteness of the model is reminiscent of the old expectation that in noncommutative spaces the theory might be free from the divergences caused by the short distance behavior of physical quantities. In this sense noncommutative theories based on compact groups resemble ordinary (commutative theories) with a momentum cutoff. It would be interesting to mention the fate of the UV/IR mixing phenomena [20]. As a generic property of models defined on canonical noncommutative spaces, see (1), certain combinations of external

momenta and the noncommutativity parameter θ may appear as a dynamical cutoff in momentum space. For example, in two-external leg diagrams of ϕ^4 theory, the combination $(p\circ p)^{-1/2}$ with $p\circ p := (p^\mu \theta^2_{\mu\nu} p^\nu)$ acts as a cutoff, causing the contribution of the so called non-planar diagram to be UV-finite [20]. In the extreme IR limit of external momenta ($p\to 0$), this cutoff tends to infinity and the result diverges. In such a case, in the IR limit of the theory the UV-divergences of the commutative (ordinary) theory are restored. This is the so called UV/IR mixing. If the noncommutative theory had been based on a commutative theory with a momentum cutoff, there would be no UV-divergence and no UV/IR mixing.

Theories discussed here are free from UV-divergences, as the momentum space is compact. In this sense, they are based on commutative theories with a momentum cutoff. Hence there is no UV-divergence in the original theory to be restored in some IR limit, and there is no room for UV/IR mixing.

3 Amplitudes

In this section the basic elements for calculation of a transition amplitude, including the construction of initial and final states, the proper normalization of the states, and the relevant kinematical factors are presented.

3.1 Fock space and initial/final states

According to the previous section, the free sector of the Lagrangian in the momentum space is given by

$$L_{\text{free}} = \frac{1}{2}\int \mathrm{d}U \;\; \Big[\dot\phi(U^{-1},t)\,\dot\phi(U,t) + \phi(U^{-1},t)\,O(U)\,\phi(U,t)\Big], \tag{66}$$

from which one obtains the canonical field momenta

$$\Pi(U,t) = \dot\phi(U^{-1},t). \tag{67}$$

The equal-time canonical commutation relations are

$$\begin{aligned} [\phi(U,t),\Pi(V,t)] &= \mathrm{i}\hbar\,\delta(U\,V^{-1}),\\ [\phi(U,t),\phi(V,t)] &= 0,\\ [\Pi(U,t),\Pi(V,t)] &= 0. \end{aligned} \tag{68}$$

As usual one might express the dynamical variables in terms of positive and negative frequency components:

$$\phi(U,t) = \sqrt{\frac{\hbar}{2\omega}}\;\big[a(U)\,\exp(-\mathrm{i}\,\omega\,t) + a^\dagger(U^{-1})\,\exp(\mathrm{i}\,\omega\,t)\big], \tag{69}$$

from which one finds

$$\begin{aligned}\left[a(U), a^\dagger(V)\right] &= \delta(UV^{-1}),\\ [a(U), a(V)] &= 0,\\ \left[a^\dagger(U), a^\dagger(V)\right] &= 0.\end{aligned} \tag{70}$$

One defines the vacuum-state through

$$\begin{aligned}a(U)\,|0\rangle &= 0, \qquad \forall\, U,\\ \langle 0|0\rangle &= 1.\end{aligned} \tag{71}$$

The multi-particle states with given momenta, being a basis of the Fock space of theory, are constructed as

$$|(U_1, n_1); (U_2, n_2); \cdots\rangle := \frac{[a^\dagger(U_1)]^{n_1}}{\sqrt{n_1!}}\,\frac{[a^\dagger(U_2)]^{n_2}}{\sqrt{n_2!}}\cdots|0\rangle. \tag{72}$$

Equetions(71) and (72) also give the normalization of multi-particle states. For example,

$$\begin{aligned}\langle U|V\rangle &= \delta(U^{-1}V),\\ \langle U|U\rangle &= \delta(\mathbf{1}).\end{aligned} \tag{73}$$

Of course, the right-hand side of the latter is infinite. But this is similar to the case of ordinary space. In the case of ordinary space, the left-hand side is finite if and only if the volume of the system is finite. In that case the left-hand side is equal to the volume of the system. One can keep the volume of the system finite and do calculations up to the point where this volume is no longer there in the observables, and then send the volume to infinity. The same thing is possible here too. In this case, instead of talking about the finiteness of the volume one takes a finite number of representations of the group. Again one does the calculations until this *volume* in the right-hand side disappears, and then sends the upper limit on the representations to infinity. The overall result is that one takes $\delta(\mathbf{1})$ as the volume of the system and deals with it like a finite number (in the intermediate stages of the calculations). In the final result, however, there should not be any $\delta(\mathbf{1})$.

3.2 *S*-matrix and transition amplitudes

An element of S-matrix, which represents the transition from the initial state i to the final state f, would come in the general form

$$S_{\mathrm{f\,i}} = \delta_{\mathrm{f\,i}} + T_{\mathrm{f\,i}}, \tag{74}$$

where the matrix elements of T come from the interaction terms. In the case of commutative space, $T_{\mathrm{f\,i}}$ contains a delta distribution corresponding to energy conservation and another delta distribution corresponding to momentum conservation. It also contains (corresponding to each incoming or outgoing particle)

a factor $\sqrt{\hbar/(2\,\omega)}$ (coming from the expression of the field in terms of creation and annihilation operators) as well as a normalization factor $\sqrt{1/\mathcal{V}}$ (where $\mathcal{V}$ is the volume of the space). One then has

$$\begin{aligned} T_{\mathrm{f\,i}} =: & 2\,\pi\,\delta\left(\sum_j \omega_{\mathrm{f}\,j} - \sum_l \omega_{\mathrm{i}\,l}\right)\,(2\,\pi)^D\,\delta\left(\sum_j \mathbf{k}_{\mathrm{f}\,j} - \sum_l \mathbf{k}_{\mathrm{i}\,l}\right) \\ & \times \prod_j \sqrt{\frac{\hbar}{2\,\omega_{\mathrm{f}\,j}\,\mathcal{V}}}\,\prod_l \sqrt{\frac{\hbar}{2\,\omega_{\mathrm{i}\,l}\,\mathcal{V}}}\;\tilde{M}_{\mathrm{f\,i}}, \end{aligned} \tag{75}$$

for ordinary space.In the case of noncommutative space, instead of $\mathcal{V}$, one has $\delta(\mathbf{1})$, and instead of the delta distribution corresponding to momentum conservation one has a delta distribution of a product of group elements corresponding to incoming and outgoing particles. Contrary to the case of ordinary space, however, the order of these group elements in the delta distribution is important. In this case one has

$$\begin{aligned} T_{\mathrm{f\,i}} =: & 2\,\pi\,\delta\left(\sum_j \omega_{\mathrm{f}\,j} - \sum_l \omega_{\mathrm{i}\,l}\right) \\ & \times \prod_j \sqrt{\frac{\hbar}{2\,\omega_{\mathrm{f}\,j}\,\delta(\mathbf{1})}}\,\prod_l \sqrt{\frac{\hbar}{2\,\omega_{\mathrm{i}\,l}\,\delta(\mathbf{1})}}\;\mathcal{M}_{\mathrm{f\,i}}, \end{aligned} \tag{76}$$

where

$$\mathcal{M}_{\mathrm{f\,i}} = \sum_\Pi M^\Pi_{\mathrm{f\,i}}\,\delta(U^\Pi). \tag{77}$$

Here U^Π is a symbolic notation meaning a product of group elements corresponding to outgoing particles, the inverse of group elements corresponding to incoming particles, and possibly group elements corresponding the loops integrated. The order of these elements is symbolically determined by Π.

$T_{\mathrm{f\,i}}$ is the amplitude of the transition. The probability of transition is the square of its modulus times the number of final states:

$$p_{\,\mathrm{i}\to\mathrm{f}} = |T_{\mathrm{f\,i}}|^2\,\prod_j [\delta(\mathbf{1})\,\mathrm{d}U_{\mathrm{f}\,j}]. \tag{78}$$

The factors $\delta(\mathbf{1})$ in the number of final states cancel the factors $\delta(\mathbf{1})$ corresponding to outgoing particles in $|T_{\mathrm{f\,i}}|^2$. There remains the factors $\delta(\mathbf{1})$ corresponding to incoming particles. In $|T_{\mathrm{f\,i}}|^2$, each term contains a product of two delta distribution of appropriate group elements, $\delta(U^\Pi)\,\delta(U^{\Pi'})$. If $(U^\Pi = \mathbf{1})$ is equivalent to $(U^{\Pi'} = \mathbf{1})$, then one can write $\delta(U^\Pi)\,\delta(U^{\Pi'})$ as $\delta(U^\Pi)\,\delta(\mathbf{1})$. This means that in $|T_{\mathrm{f\,i}}|^2$ divided by $\delta(\mathbf{1})$, only those terms survive that come from $[\delta(U^\Pi)]^2$. That is,

$$|\mathcal{M}_{\mathrm{f\,i}}|^2 \to \delta(\mathbf{1})\,\sum_\Pi |M^\Pi_{\mathrm{f\,i}}|^2\,\delta(U^\Pi). \tag{79}$$

Note the difference with the case of ordinary space. In that case one would have $|\sum_\Pi M_{\mathrm{f\,i}}|^2$ instead of $\sum_\Pi |M_{\mathrm{f\,i}}|^2$.

The rest is similar to the case of ordinary space. For a decay process, $\delta(\mathbf{1})$ in the right-hand side of (79) cancels the remaining $\delta(\mathbf{1})$ coming from the normalization of the state of the incoming particle. For a two-particle collision, one has

$$\sigma \propto p_{\mathrm{i}\to\mathrm{f}}\,\frac{1}{v_{\mathrm{rel}}\,\delta(\mathbf{1})}, \tag{80}$$

where v_{rel} is the speed of the colliding particle relative to the target, and $1/[\delta(\mathbf{1})]$ is the density of the colliding particles (one particle in a volume $\mathcal{V}$). The factor $\delta(\mathbf{1})$ in the right-hand side of the above expression cancels the remaining $\delta(\mathbf{1})$ in $|T_{\mathrm{f\,i}}|^2$, so that at the end there remains no factor of $\delta(\mathbf{1})$, as expected.

In $|T_{\mathrm{f\,i}}|^2$, there is also a term $[2\,\pi\,\delta(\omega_{\mathrm{f}}-\omega_{\mathrm{i}})]^2$, which can be written as $\mathcal{T}\,[2\,\pi\,\delta(\omega_{\mathrm{f}}-\omega_{\mathrm{i}})]$, where $\mathcal{T}$ is the interaction time, which should be sent to infinity. The transition rate is the probability divided by $\mathcal{T}$. Therefore, in the rate the factor $\mathcal{T}$ is cancelled, just as in the case of ordinary space.

These results can be summarized as

$$\mathrm{d}\Gamma = \frac{\hbar}{2\,\omega_{\mathrm{i}}}\,2\,\pi\,\delta(\omega_{\mathrm{f}}-\omega_{\mathrm{i}})\,\left[\sum_\Pi |M^\Pi_{\mathrm{f\,i}}|^2\,\delta(U^\Pi)\right]\prod_j\left(\frac{\hbar}{2\,\omega_{\mathrm{f}\,j}}\,\mathrm{d}U_{\mathrm{f}\,j}\right), \tag{81}$$

for the decay rate Γ, and

$$\mathrm{d}\sigma = \frac{1}{v_{\mathrm{rel}}}\prod_{l=1}^{2}\left(\frac{\hbar}{2\,\omega_{\mathrm{i}\,l}}\right)\,2\,\pi\,\delta(\omega_{\mathrm{f}}-\omega_{\mathrm{i}})\,\left[\sum_\Pi |M^\Pi_{\mathrm{f\,i}}|^2\,\delta(U^\Pi)\right]\prod_j\left(\frac{\hbar}{2\,\omega_{\mathrm{f}\,j}}\,\mathrm{d}U_{\mathrm{f}\,j}\right), \tag{82}$$

for the cross section σ in a two-particle collision.

Finally, let us address the relative speed v_{rel}. In the case of ordinary space, one defines the relative speed through

$$v_{\mathrm{rel}} := \sqrt{\delta^{a\,b}\,\frac{\partial\omega(\mathbf{k})}{\partial k^a}\,\frac{\partial\omega(\mathbf{k})}{\partial k^b}}, \tag{83}$$

where

$$\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2, \tag{84}$$

and $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of incoming particles. This speed does not change under exchanging particles 1 and 2, or under a rotation of the incoming momenta. In fact, as ω depends on only the length of $\mathbf{k}$, one has

$$v_{\mathrm{rel}} = \frac{\mathrm{d}\omega}{\mathrm{d}|\mathbf{k}|}. \tag{85}$$

In the case of a noncommutative space, one works most conveniently with group elements instead of momenta. Instead of $\delta^{a\,b}$, one could use the matrix elements of an invariant two-form of the algebra. One could choose the coordinates so that these elements become $\delta^{a\,b}$. Instead of $(\mathbf{k}_1 - \mathbf{k}_2)$, one could use

$(U_1\,U_2^{-1})$, or $(U_2^{-1}\,U_1)$, or their inverses. Instead of differentiation with respect to k^a, one could use the action of X_a^{L} or X_a^{R}, as the left and right invariant vector fields, respectively, whose actions at the origin (the unit element of the group) is equal to differentiation with respect to k^a. So, one would have

$$v_{\mathrm{rel}} := \sqrt{\delta^{a\,b}\,\{[L_{X_a}(\omega)](U)\}\,\{[L_{X_b}(\omega)](U)\}}, \tag{86}$$

where $L_X(\omega)$ means the action (Lie derivative) of the vector field X on the function ω. As there are four choices for U and two choices for X, it seems that one should choose between eight possible definitions for the relative speed. The function ω, however, is a class function, that is

$$\omega(V\,U\,V^{-1}) = \omega(U), \tag{87}$$

as ω^2 is in fact " $-O(U)$ ". By this, together with the fact that X_a are left or right invariant, and that $\delta^{a\,b}$ is an invariant two-form, one can show that all these choices lead to the same value for v_{rel}. Even more, one can in fact substitute $X_a(\omega)$ with the partial derivative of ω with respect to k^a. Then, as ω is a function of $|\mathbf{k}| = \sqrt{\delta_{a\,b}\,k^a\,k^b}$, it is seen that (85) holds for the case of noncommutative spaces as well. In fact,

$$v_{\mathrm{rel}} = \frac{\mathrm{d}\sqrt{-O(U)}}{\mathrm{d}|\mathbf{k}|}. \tag{88}$$

3.3 Examples

In this subsection explicit expressions for the perturbative expansion of field theory amplitudes in a space with SU(2) fuzziness are discussed.

For the propagator, let us choose the representation $s = \frac{1}{2}$ in (65):

$$\check{\Delta}(\omega, \mathbf{k}) = \frac{\mathrm{i}\,\hbar}{\omega^2 - \dfrac{16}{\ell^2}\,\sin^2\dfrac{\ell\,k}{4} - m^2}. \tag{89}$$

The reason for this choice is that it is the only representation for which, on the mass shell, energy is an increasing function of momentum. By this choice, one has for the relative velocity

$$v_{\mathrm{rel}} = \frac{\frac{2}{\ell}\,\sin\frac{\ell\,k}{2}}{\sqrt{\frac{16}{\ell^2}\,\sin^2\frac{\ell\,k}{4} + m^2}}. \tag{90}$$

We consider two types of interactions, the ϕ^3 and ϕ^4 interactions, which correspond to nonzero g_3 and g_4 in (39).

3.3.1 The three-particle interaction

The fundamental vertex with three incoming legs 1, 2, and 3 is

$$V_3^{[123]} = \frac{g_3}{2\,\mathrm{i}\,\hbar}\,2\,\pi\,\delta(\omega_1+\omega_2+\omega_3)\;[\delta(U_1\,U_2\,U_3) + \delta(U_1\,U_3\,U_2)]\,. \tag{91}$$

Now consider the scattering process $1+2 \to 3+4$. At the tree level, this process occurs via three diagrams (the s-, t-, and u-channels). Each of these channels correspond to four types group element delta functions. Of the twelve group element delta functions, however, there are only six different delta functions, each appearing in two of the three channels. The overall result corresponding to (77) is then

$$\begin{aligned}\mathcal{M}_{\mathrm{f\,i}} = \left(\frac{g_3}{2\,\mathrm{i}\,\hbar}\right)^2 \{&[\check{\Delta}(\omega_\mathrm{s},\mathbf{k}_\mathrm{s}) + \check{\Delta}(\omega_\mathrm{t},\mathbf{k}_\mathrm{t})]\,\delta(U_1\,U_2\,U_4^{-1}\,U_3^{-1})\\ &+[\check{\Delta}(\omega_\mathrm{s},\mathbf{k}_\mathrm{s}) + \check{\Delta}(\omega_\mathrm{t},\mathbf{k}_\mathrm{t})]\,\delta(U_1\,U_3^{-1}\,U_4^{-1}\,U_2)\\ &+[\check{\Delta}(\omega_\mathrm{s},\mathbf{k}_\mathrm{s}) + \check{\Delta}(\omega_\mathrm{u},\mathbf{k}_\mathrm{u})]\,\delta(U_1\,U_2\,U_3^{-1}\,U_4^{-1})\\ &+[\check{\Delta}(\omega_\mathrm{s},\mathbf{k}_\mathrm{s}) + \check{\Delta}(\omega_\mathrm{u},\mathbf{k}_\mathrm{u})]\,\delta(U_1\,U_4^{-1}\,U_3^{-1}\,U_2)\\ &+[\check{\Delta}(\omega_\mathrm{t},\mathbf{k}_\mathrm{t}) + \check{\Delta}(\omega_\mathrm{u},\mathbf{k}_\mathrm{u})]\,\delta(U_1\,U_3^{-1}\,U_2\,U_4^{-1})\\ &+[\check{\Delta}(\omega_\mathrm{t},\mathbf{k}_\mathrm{t}) + \check{\Delta}(\omega_\mathrm{u},\mathbf{k}_\mathrm{u})]\,\delta(U_1\,U_4^{-1}\,U_2\,U_3^{-1})\},\end{aligned} \tag{92}$$

where,

$$\begin{aligned}\omega_\mathrm{s} &:= \omega_1 + \omega_2,\\ \omega_\mathrm{t} &:= \omega_1 - \omega_3,\\ \omega_\mathrm{u} &:= \omega_1 - \omega_4,\end{aligned} \tag{93}$$

and

$$\begin{aligned}U_\mathrm{s} &:= U_1\,U_2,\\ U_\mathrm{t} &:= U_1\,U_3^{-1},\\ U_\mathrm{u} &:= U_1\,U_4^{-1}.\end{aligned} \tag{94}$$

It is to be noted that sending ℓ to zero, while makes the propagators equal to the commutative ones, does *not* make the transition rate equal to the commutative one. The origin of this difference, as pointed out in the previous section, comes back to the way of appearance of the δ. Here, as pointed out earlier, each possible ordering of legs of a vertex or diagram comes with a different δ, except the cases that two orderings are the same up to a cyclic permutation. This is in contrast to models on ordinary space, in which all possible orderings have the common factor of one single $\delta\big(\sum \mathbf{k}_i\big)$, representing the momentum conservation in that vertex. So, in the present case, the set of available final states is larger than the corresponding set in the commutative case. As it is seen from the delta functions, for given $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, there is not only one, but there are six possible values of $\mathbf{k}_4$. In the commutative case, all these six values are the same, so that one should add the amplitudes and then square the result. In the present case, these are not the same, so that one should add the squares, as one is calculating the transition probability to different final states. The overall result in the present case, apart from a multiplicative constant, is that the ratio of terms containing a propagator squared to the terms containing the product

of two different propagator is one. The corresponding ratio in the commutative case is one half. As mentioned in the previous section, a similar observation has been made in theories defined on κ-deformed spaces [10, 11].

3.3.2 The four-particle interaction

The fundamental vertex with four incoming legs 1, 2, 3, and 4 is

$$\begin{aligned} V_4^{[1234]} = \frac{g_4}{6\,\mathrm{i}\,\hbar}\, 2\,\pi\,\delta(\omega_1+\omega_2+\omega_3+\omega_4) \\ \times [\delta(U_1\,U_2\,U_3\,U_4) + \delta(U_1\,U_2\,U_4\,U_3) + \delta(U_1\,U_3\,U_2\,U_4) \\ +\ \delta(U_1\,U_3\,U_4\,U_2) + \delta(U_1\,U_4\,U_2\,U_3) + \delta(U_1\,U_4\,U_3\,U_2)]\,. \end{aligned} \quad (95)$$

For the scattering process $1+2 \rightarrow 3+4$, at the tree level there is a single diagram. The overall result corresponding to (77) is then

$$\begin{aligned} \mathcal{M}_{\mathrm{f\,i}} = \frac{g_4}{6\,\mathrm{i}\,\hbar}\, \big[\delta(U_1\,U_2\,U_3^{-1}\,U_4^{-1}) + \delta(U_1\,U_2\,U_4^{-1}\,U_3^{-1}) + \delta(U_1\,U_3^{-1}\,U_2\,U_4^{-1}) \\ +\ \delta(U_1\,U_3^{-1}\,U_4^{-1}\,U_2) + \delta(U_1\,U_4^{-1}\,U_2\,U_3^{-1}) + \delta(U_1\,U_4^{-1}\,U_3^{-1}\,U_2)\big]\,. \end{aligned} \quad (96)$$

In above one may observe how the different ordering of legs in a vertex come with different δ, again just as the same phenomena in the κ-deformed theories [10,11].

4 Conclusion

The structure of field theory transition amplitudes in a three-dimensional space whose spatial coordinates are noncommutative and satisfy the SU(2) Lie algebra were examined. In particular, the basic notions for constructing the observables of the theory were introduced. These include multi-particle states of the theory as a basis of Fock space, an instruction for the proper normalization of the kinematical factors associated with initial and final states of observables, as well as the way one can introduce the relative velocity between the initial states, appearing in the incident flux of an observable. Subtleties related to the proper treatment of the δ-distributions in a S-matrix expansion of the theory were discussed. Explicit examples were given for the amplitudes of an interacting scalar field theory in the lowest order of perturbation theory.

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